Analytic Perturbation Theory and New Analysis of some QCD observables

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Abstract

Here, we report briefly two topics:
1) The latest version of "Analytic Perturbation Theory" (APT) devised recently for the QCD observables both in the Euclidean and Minkowskian regions.
2) Results of the APT–based calculation for some physical processes.

1 The APT — a closed theoretical scheme

The new APT version [1, 2] mutually relates two ghost–free formulations of modified perturbation expansions for observables.

The first one, initiated about two decades ago [3, 4], changes the standard power expansion in the time-like region

$$R_{\rm pt}(s) = 1 + r_{\rm pt}(s)\,; \quad r_{\rm pt}(s) = \sum_{k\geq 1} r_k\, {\alpha_{\rm s}}^k(s;f)$$

into a nonpower one $r_{\rm pt}(s) \to r_\pi(s) = \sum_{k\geq 1} d_k\, \mathfrak{A}_k(s,f)\,.$

Here, $\alpha_{\rm s}(s)$ is a common, e.g., 3–loop QCD invariant/running coupling (usually in the $\overline{\rm MS}$ scheme – see eq.(9.5a) in Ref. [5]); and $\mathfrak{A}_k$, some integral images of the $\alpha_{\rm s}$ powers:

$$\mathfrak{A}_k(s) = \mathbf{R}\left[\bar{\alpha}_s^k(Q^2)\right]\;; \quad R(s) = \frac{i}{2\pi}\int_{s-i\varepsilon}^{s+i\varepsilon}\frac{dz}{z}\,D(-z) \equiv \mathbf{R}\left[D(Q^2)\right]\,. \tag{1}$$

The operation $\mathbf{R}$ is a reverse $\mathbf{R} = [\mathbf{D}]^{-1}$ to the one defined by the "Adler relation"

$$R(s) \to D(Q^2) = Q^2\int_0^\infty \frac{ds}{(s+Q^2)^2}\,R(s) \equiv \mathbf{D}\left\{R(s)\right\} \tag{2}$$

and transforming a real function $R(s)$ of a positive (time–like) argument into a real function $D(Q^2)$ of a positive (space–like) argument.

By operation $\mathbf{R}$, one can define [3, 4, 6] the RG–invariant effective coupling $\tilde{\alpha}(s) = \mathbf{R}\left[\bar{\alpha}_s\right]$ in the time–like region. A few simple examples are in order :

— For the one–loop case with $\bar{\alpha}_s^{(1)} = \left[\beta_0\ln(Q^2/\Lambda^2)\right]^{-1}$ one has [7, 3, 6]

$$\mathbf{R}\left[\bar{\alpha}_s^{(1)}\right] = \mathfrak{A}_1^{(1)}(s) = \frac{1}{\beta_0}\left[\frac{1}{2} - \frac{1}{\pi}\arctan\frac{L}{\pi}\right]_{L>0} = \frac{1}{\beta_0\pi}\arctan\frac{\pi}{L}\,; \quad L = \ln\frac{s}{\Lambda^2}\,. \tag{3a}$$

— Square and cube of $\bar{\alpha}_s^{(1)}$ transform into simple expressions [3, 4]

$$\mathfrak{A}_2^{(1)}(s) \equiv \mathbf{R}\left[\left(\bar{\alpha}_s^{(1)}\right)^2\right] = \frac{1}{\beta_0^2\left[L^2+\pi^2\right]} \quad \text{and} \quad \mathfrak{A}_3^{(1)}(s) = \frac{L}{\beta_0^3\left[L^2+\pi^2\right]^2}\,, \tag{3b}$$

(related by differential operation $k\beta_0 \mathfrak{A}^{(1)}_{k+1} = -(d/dL)\mathfrak{A}^{(1)}_k$) which *are not powers* of $\mathfrak{A}^{(1)}_1$.

By applying $\mathbf{D}$ to $\mathfrak{A}_k(s)$ one can "try to return" to the Euclidean domain. However, instead of $\alpha_{\rm s}$ powers, we arrive at some other functions $\mathcal{A}_k(Q^2) = \mathbf{D}\,[\mathfrak{A}_k]$, analytic in the cut Q^2-plane and free of ghost singularities. At the one–loop case

$$\beta_0 \mathcal{A}^{(1)}_1(Q^2) = \frac{1}{\ln(Q^2/\Lambda^2)} - \frac{\Lambda^2}{Q^2-\Lambda^2}\,, \quad \beta_0^2 \mathcal{A}^{(1)}_2(Q^2) = \frac{1}{\ln^2(Q^2/\Lambda^2)} + \frac{Q^2\Lambda^2}{(Q^2-\Lambda^2)^2}\,, \quad \ldots \qquad (4)$$

These expressions have been first obtained by other means [8, 9] at mid–90s. The first function $\mathcal{A}_1 = \alpha_{\rm an}(Q^2)$, an *invariant Euclidean coupling*, should now be treated as a counterpart of the *invariant Minkowskian coupling* [6] $\tilde{\alpha}(s) = \mathfrak{A}_1(s)$. Both $\alpha_{\rm an}$ and $\tilde{\alpha}$ are real monotonically decreasing functions with the same maximum value $\alpha_{\rm an}(0) = \tilde{\alpha}(0) = 1/\beta_0(f=3) \simeq 1.4$ in the IR limit. All higher functions vanish, $\mathcal{A}_k(0) = \mathfrak{A}_k(0) = 0$ in this limit. For $k \geq 2$, they oscillate in the IR region.

The same properties remain valid for a higher–loop case. Explicit expressions for $\mathcal{A}_k$ and $\mathfrak{A}_k$ at the two–loop case can be written (see, Ref. [10]) in terms of a special Lambert function. They are presented in Figs 1a and 1b.

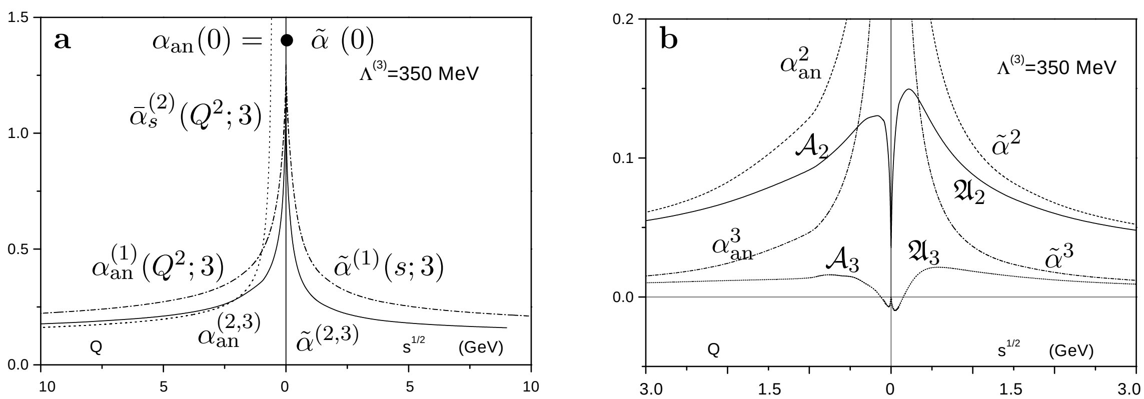

Figure 1: **a** *Space-like and time-like global analytic couplings in a few GeV domain;* **b** *"Distorted mirror symmetry" for global expansion functions. All the curves in* **1b** *are given for the 2–loop case.*

Here, in Fig.1a, by the dotted line we give a usual two-loop effective QCD coupling $\bar{\alpha}_s(Q^2)$ with a pole at $Q^2 = \Lambda^2$. On the other hand, the dash–dotted curves represent the one-loop APT approximations (3a) and (4). The solid APT curves are based on the exact two-loop solutions of RG equations and approximate three–loop solutions in the $\overline{\rm MS}$ scheme. Their remarkable coincidence (within the 1–2 per cent) demonstrates reduced sensitivity of the APT with respect to higher–loops effects in the whole Euclidean and Minkowskian regions from IR to UV limits. Fig.1b shows higher functions calculated at the two–loop case.

Remarkably enough, the mechanism of liberation of unphysical singularities is quite different. While in the space-like domain it involves nonperturbative, power in Q^2, structures, in the time-like region, it is based only upon resummation of the "π^2 terms". Figuratively, (non-perturbative !) *analyticization* [11] in the Q^2–channel can be treated as a quantitatively distorted reflection (under $Q^2 \to s = -Q^2$) of (perfectly perturbative) π^2–resummation in the s–channel. This effect of "distorting mirror" first discussed in [12] is clearly seen in figures.

In a real case, the procedure of the threshold matching is in use. E.g., in the $\overline{\rm MS}$ scheme with $\bar\alpha_s(Q^2=M_f^2;f-1)=\bar\alpha_s(Q^2=M_f^2;f)$ it defines a "global" function

$$\bar\alpha_s(Q^2)=\bar\alpha_s(Q^2;f) \quad \text{at} \quad M_{f-1}^2\leq Q^2\leq M_f^2\,,$$

continuous in the space-like region of positive Q^2 values with discontinuity of derivatives at matching points. To this there corresponds a discontinuous spectral density

$$\rho_k(\sigma)=\rho_k(\sigma;3)+\sum_{f\geq4}\theta(\sigma-M_f^2)\left\{\rho_k(\sigma;f)-\rho_k(\sigma;f-1)\right\} \tag{5}$$

with $\rho_k(\sigma;f)=\Im\,\bar\alpha_s^k(-\sigma,f)$ which yields the smooth global Euclidean and spline–continuous global Minkowskian expansion functions

$$\mathcal{A}_k(Q^2)=\frac{1}{\pi}\int\limits_0^\infty\frac{d\sigma}{\sigma+x}\,\rho_k(\sigma)\,;\quad \mathfrak{A}_k(s)=\int\limits_s^\infty\frac{d\sigma}{\sigma}\rho_k(\sigma)\,. \tag{6}$$

2 The APT applications.

To illustrate a qualitative difference between our global APT scheme and common practice of data analysis, we consider a few examples.

In the usual treatment — see, e.g., Ref. [5] — the (QCD perturbative part of) Minkowskian observable, like e^+e^- annihilation or Z_0 decay, is presented in the form

$$\frac{R(s)}{R_0}=1+r(s)\,;\quad r_{PT}(s)=\frac{\bar\alpha_s(s)}{\pi}+r_2\,\bar\alpha_s^2(s)+r_3\,\bar\alpha_s^3(s)\,. \tag{7}$$

Here, coefficients $r_1=1/\pi\,,\ r_2$ and r_3 usually are not diminishing. A rather big negative r_3 value comes mainly from the $-r_1\pi^2\beta_0^2/3$ term. In the APT, we have instead

$$r_{APT}(s)=d_1\tilde\alpha(s)+d_2\,\mathfrak{A}_2(s)+d_3\,\mathfrak{A}_3(s) \tag{8}$$

with reasonably decreasing coefficients $d_1=r_1,\ d_2=r_2$ and $d_3=r_3+r_1\pi^2\beta_0^2/3$, the mentioned π^2 term of r_3 being "swallowed" by $\tilde\alpha(s)$.

In the Euclidean channel, instead of a power expansion similar to (7), we typically have

$$d_{APT}(Q^2)=d_1\alpha_{\rm an}(Q^2)+d_2\,\mathcal{A}_2(Q^2)+d_3\,\mathcal{A}_3(Q^2)\,. \tag{9}$$

Here, the modification is related to nonperturbative structures like in (4).

Table 1 : Relative contributions (in %) of 1– , 2– and 3–loop terms to observables

Process	*PT*			*APT*		
GLS sum rule	65	24	11	75	21	**4**
Bjorken. s.r.	55	26	19	80	19	**1**
Incl. τ-decay	55	29	16	88	11	**1**
$e^+e^-\to$ hadr. (at 10 GeV)	96	8	-4	92	7	**.5**
$Z_o\to$ hadr.	98.6	3.7	-2.3	96.9	3.5	**-.4**

In Table 1, we give values of the relative contribution of the first, second, and third terms of the r.h.s. in (7),(8) and (9) for Gross–Llywellin-Smith [13] and Bjorken [14] sum rules, τ – decay in the vector channel [15] as well as for e^+e^- and Z_0 inclusive cross-sections. As it follows from this Table, in the APT case, the three–loop (last) term is very small, and being compared with data errors, numerically unessential. This means that, in practice,

one can use the APT expansions (8) and (9) without the last term.

Using this conclusion as a hint, we reanalyzed data in the five–flavor region. Results are presented in Table 2.

Table 2 : The APT revised part ($f = 5$) of Bethke's [16] Table 6

Figures in brackets give the difference in the last digit between APT and common $\bar{\alpha}_s(M_Z^2)$ values.

Process	$\sqrt{s}$ GeV	loops No	$\bar{\alpha}_s$ (s) ref.[2]	$\bar{\alpha}_s(M_Z^2)$ ref.[2]	$\bar{\alpha}_s$ (s) APT	$\bar{\alpha}_s(M_Z^2)$ APT
Υ-decay [5]	9.5	2	.170	.114	.182	.120 (+6)
$e^+e^-[\sigma_{had}]$	10.5	**3**	.200	.130	.198	.129(**-1**)
$e^+e^-[j\ \&\ sh]$	22.0	2	.161	.124	.166	.127(+3)
$e^+e^-[j\ \&\ sh]$	35.0	2	.145	.123	.149	.126(+3)
$e^+e^-[\sigma_{had}]$	42.4	**3**	.144	.126	.145	.127(+**1**)
$e^+e^-[j\ \&sh]$	44.0	2	.139	.123	.142	.126(+3)
$e^+e^-[j\ \&sh]$	58	2	.132	.123	.135	.125(+2)
$Z_0 \to$ **had.**	91.2	**3**	.124	.124	.124	.124 (**0**)
$e^+e^-[j\ \&\ sh]$	91.2	2	.121	.121	.123	.123(+2)
-"-	....	2	...	...	...	... (+2)
$e^+e^-[j\ \&\ sh]$	189	2	.110	.123	.112	.125(+2)

Averaged $< \bar{\alpha}_s(M_Z^2) >_{f=5}$ values = 0.121 0.124 .

Addressing the reader interested in a more detail to our recent paper [17], we shortly comment that transition PT $\to$ APT from the standard algorithm to our new one for the NLO case, as a rule, enlarges extracted $\bar{\alpha}_s(M_Z^2)$ values by 0.002 (or more) which results in the averaged $\bar{\alpha}_s(M_Z^2)$ value equal to 0.124.

At the same time it improves the correlation of events in the $f = 5$ region. More specifically, it changes the $< \chi^2 >_{f=5}$ value from 0.197 to 0.144.

3 Summary.

1. First, we have outlined a recently devised self-consistent scheme for analyzing data both in the space-like and time-like regions. Within this APT scheme, perturbative expressions for an observable involve expansions over the sets $\{\mathcal{A}_k(Q^2)\}$ and $\{\mathfrak{A}_k(s)\}$, that are nonpower series, free of unphysical singularities, with usual numerical coefficients d_k obtained by calculation of the relevant Feynman diagrams.

2. Numerically, the APT calculations reveal reduced sensitivity to the NNLO effects, as it has been demonstrated in Table 1.

Table 2 summarizes our attempt to “improve” some particular data for $\bar{\alpha}_s(M_Z^2)$ values extracted from experiments in the Minkowskian five-flavour region. These results look encouraging. In particular, they yield the new value

$$< \bar{\alpha}_s(M_Z^2) >_{f=5} = 0.124\,,$$

quite different from the widely accepted “world average” 0.118 and even from the usual average (= 0.121) over the five–flavour region.

3. This result, being taken as granted, rises a physical question on mutual consistency of current data on the QCD–invariant coupling behavior in the “medium ($f = 3, 4$)” and “high ($f = 5, 6$)” regions.

Answer to this question could be obtained by a further revised (within the APT technique) calculation.

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